Lesson 2: Inscribed Polygons and Circumscribed Triangles

Common Core Georgia Performance Standard
MCC9–12.G.C.3

Essential Questions
1. What is the purpose of locating the incenter?
2. What is the purpose of locating the circumcenter?
3. Where are an incenter and an inscribed circle used in real life?
4. What types of quadrilaterals can be inscribed in a circle?

WORDS TO KNOW
angle bisector        a ray that divides an angle into two congruent angles
circumcenter          the intersection of the perpendicular bisectors of a triangle
circumscribed circle  a circle that contains all vertices of a polygon
circumscribed triangle triangle whose sides are tangent to an interior circle
equidistant           a point or points that lie the same distance away from a given object
incenter              the intersection of the angle bisectors of a triangle
inscribed circle      a circle whose tangents form a triangle
inscribed quadrilateral a quadrilateral whose vertices are on a circle
inscribed triangle    a triangle whose vertices are tangent to a circle
perpendicular bisector a segment that is perpendicular to a given segment and contains the midpoint of that segment
point of concurrency   the point where three or more lines intersect
Recommended Resources

  
  http://www.walch.com/rr/00051

  This website describes how to construct the incenter in The Geometer’s Sketchpad dynamic geometry program.

- IXL Learning. “Angles in inscribed quadrilaterals.”
  
  http://www.walch.com/rr/00052

  This interactive website gives a series of problems and scores them immediately. If the user submits a wrong answer, a description and process for arriving at the correct answer are provided. This activity is meant as practice for inscribed polygons.

- IXL Learning. “Identify medians, altitudes, angle bisectors, and perpendicular bisectors.”
  
  http://www.walch.com/rr/00053

  This interactive website gives a series of problems and scores them immediately. If the user submits a wrong answer, a description and process for arriving at the correct answer are provided. These problems start with a triangle and a construction of a point of concurrency. Users are required to choose which point of concurrency is depicted. This activity is meant as a review of key terms for inscribed polygons and circumscribed triangles.
Lesson 3.2.1: Constructing Inscribed Circles

Introduction

In the map of Georgia below, Interstates 475 and 75 form a triangle with Macon as one of the vertices. If a company wants to build a distribution center in the middle of that triangle so that the building will be equidistant from each interstate, where should the distribution center be built? In this lesson, we will investigate the point that solves this problem and the geometry that supports it.

Key Concepts

- A point that is **equidistant** lies the same distance away from a given object.
- The company wants its distribution center to be equidistant from the given interstates.
- To determine the location of the distribution center, the company would first need to determine the point at which the distribution center would be equidistant from each of the interstates.
- To determine this point, the company would need to find the angle bisectors of the triangle created by the interstates.
• An **angle bisector** is the ray that divides an angle into two congruent angles.
• When all three angle bisectors of a triangle are constructed, the rays are said to be concurrent, or intersect at one point.
• This point is called the **point of concurrency**, where three or more lines intersect.
• The point of concurrency of the angle bisectors of a triangle is called the **incenter**.
• The incenter is equidistant from the three sides of the triangle and is also the center of the circle tangent to the triangle.
• A circle whose tangents form a triangle is referred to as an **inscribed circle**.
• When the inscribed circle is constructed, the triangle is referred to as a **circumscribed triangle** — a triangle whose sides are tangent to a circle.

To construct an inscribed circle, determine the shortest distance from the incenter to each of the sides of the triangle.
• Remember that the shortest distance from a point to a line or line segment is a perpendicular line.
• Follow the steps outlined in the Guided Practice that follows to construct a line perpendicular to a point not on the line.
Guided Practice 3.2.1

Example 1

Verify that the angle bisectors of acute \( \triangle ABC \) are concurrent and that this concurrent point is equidistant from each side.

1. Construct the angle bisector of \( \angle A \).

First, place the compass on point \( A \) and swing an arc that intersects the two sides of the angle.
From each point of intersection, swing an arc in the interior of the angle. These two arcs should intersect. If they do not, increase the radius of the arc. Connect $A$ with the intersections of the two arcs to locate the angle bisector.

2. Repeat the process for $\angle B$ and $\angle C$. 

![Diagram showing the process for finding the angle bisector between points A, B, and C.]
3. Locate the point of concurrency. Label this point as $D$.

The point of concurrency is where all three of the angle bisectors meet. As seen on the coordinate plane, the point of concurrency is $(2, 1)$.

4. Verify that the point of concurrency is equidistant from each side.

Determine the line that is perpendicular to $\overline{AC}$ through point $D$ $(2, 1)$.

Find the equation of the line representing $\overline{AC}$.

Use the slope formula to calculate the slope of the line.

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

Slope formula

\[
m = \frac{(-7) - (7)}{(-2) - (-2)}
\]

Substitute $(-2, 7)$ and $(-2, -7)$ for $(x_1, y_1)$ and $(x_2, y_2)$.

\[
m = \frac{-14}{0}
\]

The slope of $\overline{AC}$ is undefined because the line is vertical.

(continued)
The equation of a vertical line always has the form $y = a$, where $a$ is the $x$-intercept. In this case, the $x$-intercept is $-2$.

The equation of $\overline{AC}$ is $y = -2$.

The line that is perpendicular to $y = -2$ is horizontal and of the form $x = b$, where $b$ is the $y$-intercept.

The line that is horizontal at $(2, 1)$ intersects the $y$-intercept at 1.

The equation of the line that is perpendicular to $\overline{AC}$ is $x = 1$.

The intersection of the two lines is $(-2, 1)$. Label this point $E$.

Use the distance formula to calculate the length of $\overline{DE}$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Distance formula

$$\sqrt{[(2) - (-2)]^2 + [(1) - (1)]^2}$$

Substitute $(-2, 1)$ and $(2, 1)$ for $(x_1, y_1)$ and $(x_2, y_2)$.

$$DE = \sqrt{4^2 + 0^2}$$

Simplify.

$$DE = \sqrt{16}$$

$$DE = 4$$

(continued)
Determine the line that is perpendicular to $\overline{AB}$ through point $D(2, 1)$.

Find the equation of the line representing $\overline{AB}$.

The slope of $\overline{AB}$ is $\frac{5}{12}$.

Use the point-slope form of a linear equation, $y = mx + b$, to determine the equation of $\overline{AB}$.

The equation of $\overline{AB}$ is $y = \frac{5}{12}x + \frac{37}{6}$.

The line that is perpendicular to $\overline{AB}$ has a slope that is the opposite reciprocal of the slope of $\overline{AB}$.

The opposite reciprocal of $\frac{5}{12}$ is $\frac{12}{5}$.

Again, use the point-slope form of a linear equation to determine the perpendicular line.

The equation of the perpendicular line is $y = \frac{12}{5}x - \frac{19}{5}$.

The intersection of the two lines can be found by setting the equations equal to each other and solving for $x$.

The intersection of the two lines is at $\left(\frac{46}{13}, \frac{61}{13}\right)$. Label this point $F$. 

(continued)
Use the distance formula to calculate the length of \( DF \).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

Distance formula

\[
DF = \sqrt{\left(2 - \frac{46}{13}\right)^2 + \left(1 - \frac{61}{13}\right)^2}
\]

Substitute \( \left(\frac{46}{13}, \frac{61}{13}\right) \) and \( (2, 1) \) for \( (x_1, y_1) \) and \( (x_2, y_2) \).

\[
DF = \sqrt{\left(\frac{26}{13} - \frac{46}{13}\right)^2 + \left(\frac{13}{13} - \frac{61}{13}\right)^2}
\]

Simplify.

\[
DF = \sqrt{2.37 + 13.63}
\]

\[
DF = \sqrt{16}
\]

\[
DF = 4
\]

By following these same steps, it can be determined that the point of intersection of \( BC \) and the line that is perpendicular to \( BC \) through \( D(2, 1) \) is at \( \left(\frac{22}{5}, \frac{11}{5}\right) \).

The midpoint of \( BC \) is at \((4.4, -2.2)\). Label this point \( G \).

(continued)
Use the distance formula to calculate the length of $DG$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Distance formula

$$DG = \sqrt{\left[(2) - \left(\frac{22}{5}\right)\right]^2 + \left[(1) - \left(-\frac{11}{5}\right)\right]^2}$$

Substitute $\left(\frac{22}{5}, -\frac{11}{5}\right)$ and $(2, 1)$ for $(x_1, y_1)$ and $(x_2, y_2)$.

Simplify.

$$DG = \sqrt{16}$$

$DG = 4$

5. State your conclusion.

The angle bisectors of the triangle are concurrent at point $D$. The three segments perpendicular from point $D$ (the incenter of the triangle) to the sides of the triangle are equal; therefore, the incenter is equidistant from the three sides of the triangle.
Example 2

Construct a circle inscribed in acute $\triangle ABC$.

1. Construct the angle bisectors of each angle.
   Use the same method described in Example 1 to bisect each angle.
   Label the point of concurrency of the angle bisectors as point $D$.

2. Construct a perpendicular line from $D$ to each side.
   To construct a perpendicular line through $D$, put the sharp point of your compass on point $D$. Open the compass until it extends farther than $\overline{AC}$.
   Make a large arc that intersects $\overline{AC}$ in exactly two places. Without changing your compass setting, put the sharp point of the compass on one of the points of intersection. Make a second arc below the given line.
   Without changing your compass setting, put the sharp point of the compass on the second point of intersection. Make a third arc below the given line. The third arc must intersect the second arc.
   Label the point of intersection $E$.
   Use your straightedge to connect points $D$ and $E$.
   Repeat this process for each of the remaining sides.
3. Verify that the lengths of the perpendicular segments are equal.
   Use your compass and carefully measure the length of each perpendicular segment.
   Verify that the measurements are the same.

![Diagram of triangle and perpendicular segments]

4. Construct the inscribed circle with center $D$.
   Place the compass point at $D$ and open the compass to the length of any one of the perpendicular segments.
   Use this setting to construct a circle inside of $\triangle ABC$.

![Diagram of inscribed circle]

Circle $D$ is inscribed in $\triangle ABC$ and is tangent to each of the sides of the triangle.
**Example 3**

Construct a circle inscribed in obtuse $\triangle ABC$.

1. Construct the angle bisectors of each angle.
   
   Use the same method as described in Example 1 to bisect each angle. Label the point of concurrency of the angle bisectors as point $D$. 
2. Construct a perpendicular line from $D$ to each side.

To construct a perpendicular line through $D$, put the sharp point of your compass on point $D$. Open the compass until it extends farther than $AC$. Make a large arc that intersects $AC$ in exactly two places. Without changing your compass setting, put the sharp point of the compass on one of the points of intersection. Make a second arc below the given line.

Without changing your compass setting, put the sharp point of the compass on the second point of intersection. Make a third arc below the given line. The third arc must intersect the second arc.

Label the point of intersection $E$.

Use your straightedge to connect points $D$ and $E$.

Repeat this process for each of the remaining sides, labeling the points of intersection $F$ and $G$. 
3. Verify that the lengths of the perpendicular segments are equal.
   Use your compass and carefully measure the length of each perpendicular.
   Verify that the measurements are the same.

4. Construct the inscribed circle with center $D$.
   Place the compass point at $D$ and open the compass to the length of any one of the perpendicular segments.
   Use this setting to construct a circle inside of $\triangle ABC$.

Circle $D$ is inscribed in $\triangle ABC$ and is tangent to each of the sides of the triangle.
Practice 3.2.1: Constructing Inscribed Circles

Construct the inscribed circle for each of the triangles in problems 1–3.

1.

2.

3.

4. Will the incenter ever be located outside of the triangle? Why or why not?
Consider your constructions of the previous three problems.
5. In the map of Georgia below, Interstates 16 and 75 and Route 280 form a triangle with Macon as one of the vertices. A company wants to build its new headquarters in the middle of that triangle so that the building is equidistant from each highway. Where should the headquarters be built?

6. Find the length of $BI$. Assume that $I$ is the incenter.
7. Find the measure of $\angle BIC$. Assume that $BI$ and $CI$ are angle bisectors.

8. Find the measure of $\angle AIB$. Assume that $BI$ and $AI$ are angle bisectors.

continued
9. Suppose that $\triangle ABC$ is equilateral and point $I$ is the incenter. What is true about $\triangle ABI$ and $\triangle ACI$? Support your answer.

![Diagram of an equilateral triangle with point I as the incenter]

10. Given the circle shown, construct a circumscribed triangle.

![Diagram of a circle with a circumscribed triangle]
Lesson 3.2.2: Constructing Circumscribed Circles

Introduction

The owners of a radio station want to build a new broadcasting building located within the triangle formed by the cities of Atlanta, Columbus, and Macon. Where should the station be built so that it is equidistant from each city? In this lesson, we will investigate the point that solves this problem and the geometry that supports it.

Key Concepts

- To determine the location of the broadcasting building, the station owners would first need to determine the point at which the building would be equidistant from each of the three cities.
- To determine this point, the owners would need to find the perpendicular bisector of each of the sides of the triangle formed by the three cities.
- The perpendicular bisector of a segment is the segment that is perpendicular to a given segment and contains the midpoint of that segment.
• When all three perpendicular bisectors of a triangle are constructed, the rays intersect at one point.

• This point of concurrency is called the **circumcenter**.

• The circumcenter is equidistant from the three vertices of the triangle and is also the center of the circle that contains the three vertices of the triangle.

• A circle that contains all the vertices of a polygon is referred to as the **circumscribed circle**.

• When the circumscribed circle is constructed, the triangle is referred to as an **inscribed triangle**, a triangle whose vertices are tangent to a circle.

• To construct the circumscribed circle, determine the shortest distance from the circumcenter to each of the vertices of the triangle.

• Remember that the shortest distance from a point to a line or line segment is a perpendicular line.

• Follow the steps in the Guided Practice examples that follow to construct a perpendicular bisector in order to find the circumcenter of the triangle.
Example 1
Verify that the perpendicular bisectors of acute \( \triangle ABC \) are concurrent and that this concurrent point is equidistant from each vertex.

1. Construct the perpendicular bisector of \( \overline{AB} \).
   
   Set the compass at point \( A \) and swing an arc.
   
   Using the same radius, swing an arc from point \( B \) so that the two arcs intersect.
   
   If they do not intersect, lengthen the radius for each arc.
   
   Connect the two intersections to locate the perpendicular bisector.
2. Repeat the process for $\overline{BC}$ and $\overline{AC}$.

![Diagram showing construction of perpendicular bisectors]

3. Locate the point of concurrency. Label this point $D$.
   The point of concurrency is where all three perpendicular bisectors meet.

![Diagram showing point of concurrency $D$]

4. Verify that the point of concurrency is equidistant from each vertex.
   Use your compass and carefully measure the length from point $D$ to each vertex.
   The measurements are the same.
Example 2
Construct a circle circumscribed about acute \( \triangle ABC \).

1. Construct the perpendicular bisectors of each side.
   Use the same method as described in Example 1 to bisect each side.
   Label the point of concurrency of the perpendicular bisectors as point \( D \).
2. Verify that the point of concurrency is equidistant from each of the vertices.

Use your compass and carefully measure the length of each perpendicular from \( D \) to each vertex.

Verify that the measurements are the same.

3. Construct the circumscribed circle with center \( D \).

Place the compass point at \( D \) and open the compass to the distance of \( D \) to any vertex.

Use this setting to construct a circle around \( \triangle ABC \).

Circle \( D \) is circumscribed about \( \triangle ABC \) and each vertex is tangent to the circle.
Example 3

Construct a circle circumscribed about obtuse $\triangle ABC$.

1. Construct the perpendicular bisectors of each side.
   Use the same method as described in Example 1 to bisect each side.
   Label the point of concurrency of the perpendicular bisectors as point $D$. 
2. Verify that the point of concurrency is equidistant from each of the vertices.

   Use your compass and carefully measure the length of each perpendicular from $D$ to each vertex.

   Verify that the measurements are the same.

3. Construct the circumscribed circle with center $D$.

   Place the compass point at $D$ and open the compass to the distance of $D$ to any vertex.

   Use this setting to construct a circle around $\triangle ABC$.

Circle $D$ is circumscribed about $\triangle ABC$ and each vertex is tangent to the circle.
Practice 3.2.2: Constructing Circumscribed Circles

Construct the circumscribed circle for each of the triangles in problems 1–3.

1. [Diagram of triangle 1]

2. [Diagram of triangle 2]

3. [Diagram of triangle 3]
Lesson 2: Inscribed Polygons and Circumscribed Triangles

Use what you’ve learned and the diagrams, when provided, to complete problems 4–10.

4. Where is the circumcenter in a right triangle? Is this true for all right triangles?

5. Jane, Keith, and Lee are shopping at a mall. The halls are arranged in a star shape, as shown in the diagram. The friends’ locations are marked on the diagram. Through texting, they arrange to meet up so they can grab a cinnamon bun. Where should they meet so that each person has the same distance to walk?

6. Assume that point C is the circumcenter for $\triangle ABD$. What is the length of $AB$?
7. In planning a new technology building for a college, an architect needs to make sure that the server for the computer network will be in a room that is equidistant from three computer labs. In which room should the server be placed?

8. Can the circumcenter of a triangle ever be located outside of the triangle? Explain your reasoning.

9. Is it possible for the incenter to be the same point as the circumcenter? Why or why not? If it is possible, what type(s) of triangle would fit this criterion? Consider your responses to problems 1 through 3 in determining your answer.

10. Describe a method to verify the center of the circle below. Then, carry out your plan.
Lesson 3.2.3: Proving Properties of Inscribed Quadrilaterals

Introduction

One of the most famous drawings of all time is Leonardo da Vinci’s Vitruvian Man. Da Vinci’s sketch was of a man enclosed by a circle that touched the man’s feet and hands. In this lesson, we will investigate the properties of quadrilaterals inscribed in a circle.
Key Concepts

- An **inscribed quadrilateral** is a quadrilateral whose vertices are on a circle.
- The opposite angles of an inscribed quadrilateral are supplementary.

\[
\begin{align*}
\angle A + \angle C &= 180 \\
\angle B + \angle D &= 180
\end{align*}
\]

- Remember that the measure of an inscribed angle is half the measure of the intercepted arc.
- Rectangles and squares can always be inscribed within a circle.
Guided Practice 3.2.3

Example 1

Consider the inscribed quadrilateral in the following diagram. What are the relationships between the measures of the angles of an inscribed quadrilateral?

1. Find the measure of $\angle B$.

$\angle B$ is an inscribed angle. Therefore, its measure will be equal to half the measure of the intercepted arc.

The intercepted arc $\widehat{ADC}$ has a measure of $122 + 52$, or $174^\circ$.

The measure of $\angle B$ is $\frac{1}{2}$ of $174$, or $87^\circ$. 
2. Find the measure of $\angle D$.

The intercepted arc $\overline{ABC}$ has a measure of $82 + 104$, or $186^\circ$.

The measure of $\angle D$ is $\frac{1}{2}$ of $186$, or $93^\circ$.

3. What is the relationship between $\angle B$ and $\angle D$?

Since the sum of the measures of $\angle B$ and $\angle D$ equals $180^\circ$, $\angle B$ and $\angle D$ are supplementary angles.

4. Does this same relationship exist between $\angle A$ and $\angle C$?

The intercepted arc $\overline{BCD}$ has a measure of $104 + 52$, or $156^\circ$.

The measure of $\angle A$ is $\frac{1}{2}$ of $156$, or $78^\circ$.

The intercepted arc $\overline{BAD}$ has a measure of $82 + 122$, or $204^\circ$.

The measure of $\angle C$ is $\frac{1}{2}$ of $204$, or $102^\circ$.

The sum of the measures of $\angle A$ and $\angle C$ also equals $180^\circ$; therefore, $\angle A$ and $\angle C$ are supplementary.

5. State your conclusion.

The opposite angles of an inscribed quadrilateral are supplementary.
**Example 2**

Consider the inscribed quadrilateral below. Do the relationships discovered between the angles in Example 1 still hold for the angles in this quadrilateral?

1. Calculate the measures of all four angles of quadrilateral $ABCE$.

   $\angle A$ intercepts $BCE$, so the measure of $\angle A$ is half the measure of $BCE$.
   
   $m\angle A = \frac{1}{2}(104 + 74) = 89$

   $\angle B$ intercepts $AEC$, so the measure of $\angle B$ is half the measure of $AEC$.
   
   $m\angle B = \frac{1}{2}(100 + 74) = 87$

   $\angle C$ intercepts $BAE$, so the measure of $\angle C$ is half the measure of $BAE$.
   
   $m\angle C = \frac{1}{2}(100 + 82) = 91$

   $\angle E$ intercepts $ABC$, so the measure of $\angle E$ is half the measure of $ABC$.
   
   $m\angle E = \frac{1}{2}(104 + 82) = 93$
2. Find the sum of the measures of $\angle A$ and $\angle C$.
   The sum of the measures of $\angle A$ and $\angle C$ is equal to $89 + 91 = 180$.

3. State your conclusion.
   The measures of $\angle A$ and $\angle C$ sum to $180^\circ$, as do the measures of $\angle B$ and $\angle E$; therefore, it is still true that opposite angles of an inscribed quadrilateral are supplementary.

Example 3
Prove that the opposite angles of the given inscribed quadrilateral are supplementary.

1. What is the sum of $w^\circ + x^\circ + y^\circ + z^\circ$?
   Together, the arcs create a circle that measures $360^\circ$; therefore, the sum of the arc measures is $360$. 

2. Find the measure of each angle of quadrilateral $ABCD$.

\[ m \angle A = \frac{1}{2}(x + y) \quad m \angle C = \frac{1}{2}(w + z) \]
\[ m \angle B = \frac{1}{2}(x + z) \quad m \angle D = \frac{1}{2}(w + x) \]

3. Find the sum of the measures of $\angle A$ and $\angle C$.

\[ m \angle A + m \angle C = \frac{1}{2}(x + y) + \frac{1}{2}(w + z) \]
\[ = \frac{1}{2}(x + y + w + z) \]
\[ = \frac{1}{2}(360) = 180 \]

4. Find the sum of the measures of $\angle B$ and $\angle D$.

\[ m \angle B + m \angle D = \frac{1}{2}(y + z) + \frac{1}{2}(w + x) \]
\[ = \frac{1}{2}(y + z + w + z) \]
\[ = \frac{1}{2}(360) = 180 \]

5. State your conclusion.

$\angle A + \angle C = 180$ and $\angle B + \angle D = 180$. Therefore, each pair of opposite angles of an inscribed quadrilateral is supplementary.
Practice 3.2.3: Proving Properties of Inscribed Quadrilaterals

Use the provided diagrams and your knowledge of the properties of inscribed quadrilaterals to complete problems 1–5.

1. Find the values of \( x \) and \( y \).

2. Find the values of \( k \) and \( n \).
3. Are any two sides of the inscribed quadrilateral parallel? If so, which? Support your answer.

4. Construct a square inscribed in the innermost circle. Is the new square similar to the original square? Why or why not?
5. Is the quadrilateral below a parallelogram? Why or why not?

Use the figure below to complete problems 6–8.

6. Find the measure of \( \overarc{AE} \).

7. Find the measure of \( \angle A \).

8. Find the measure of \( \angle D \).
Use your knowledge of inscribed quadrilaterals to complete problems 9 and 10.

9. Is it possible for a kite to be inscribed in a circle? Why or why not?

10. If a rectangle is inscribed in a circle, each diagonal also serves another function. What is this function?